Complex analysis for EE, 2012-13, problem set 7

1. Suppose a function f is holomorphic on $\mathbb{C}\setminus\{z_1, z_2, \ldots, z_n\}$. Show that there exist entire functions $f_0, f_1, \ldots, f_n$ such that $f(z) = f_0(z) + f_1(\frac{1}{z-z_1}) + \ldots + f_n(\frac{1}{z-z_n})$.

2. Suppose f is holomorphic in $\mathbb{C}$, except perhaps at finitely many poles, and either has a pole at infinity or the limit $\lim_{z\to\infty} f(z)$ exists. Such a function is called *meromorphic* on the Riemann sphere. Show that f is a rational function (hint: isolate the poles of f to arrive by an entire function g such that $f(z) = g(z)/\prod_{j=1}^{n}(z-z_j)^{m_j}$. Next, what can you say about g's zeroes?)

3. Let Ω be a bounded region, and let $F \subseteq \Omega$ be closed. Show that there does not exist a function f with infinitely many poles in F that otherwise holomorphic on Ω.

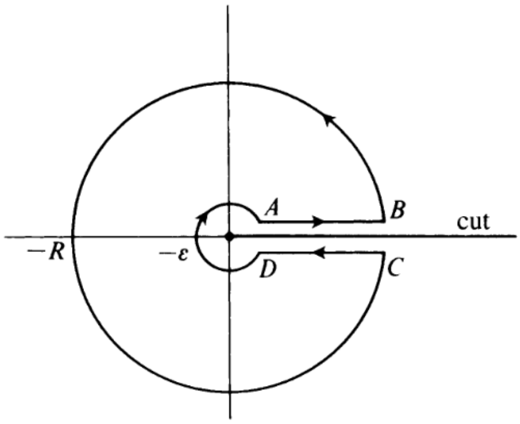
4. Find the residues at the poles of the following functions:
   (a) $\frac{1}{z^3(z^2+1)}$
   (b) $\frac{1}{z(3-z)}$
   (c) $\frac{1}{1-z+z^2}$
   (d) $\frac{1}{(z^2+z+1)^3}$
   (e) $\frac{e^{iz}}{\cosh(z)}$
   (f) $\frac{1}{(z^2+1)\sin(\pi z)}$
   (g) $\frac{1}{z^4\sin(z)}$
   (h) $\tan(z)^2$

5. By converting each integral into one along the unit circle, prove the following identities:
   (a) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{4}$
   (b) $\int_0^{2\pi} \frac{1}{1+a\cos\theta}d\theta = \frac{2\pi}{\sqrt{1-a^2}}$, where $-1 < a < 1$
   (c) $\int_0^{2\pi} \frac{1}{(1+2a\cos\theta+a^2)^2}d\theta = \frac{2\pi(1+a^2))}{(1-a^2)^3}$, where $-1 < a < 1$

6. By integrating along a rectangle which rests upon the real axis, show that $\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}dx = \frac{\pi}{\sin(\pi\alpha)}$, where $0 < \alpha < 1$.

7. Evaluate the following integrals
   (a) $\int_0^{\infty} \frac{dx}{x^{10}+x^5+1}$.
   (b) $\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^4+x^3+x^2+x+1}dx$

8. A keyhole contour is described in the following image:



In this next exercise, we take $R = 1$, and suppose f is holomorphic in some neighborhood of the closed unit disk. We take $\log z$ to be the analytic branch of logarithm in $\mathbb{C} \setminus [0, \infty)$.

(a) Show that

$$\int_{AB} f(z)(\log z - i\pi)dz = \int_{\varepsilon}^{1} f(x+i\delta)(\log(x+i\delta)-\pi i)dx \underset{\delta\to 0}{\longrightarrow} \int_{\varepsilon}^{1} f(x)(\log(x)-\pi i)dx,$$

and that

$$\int_{CD} f(z)(\log z - i\pi)dz = -\int_{\varepsilon}^{1} f(x-i\delta)(\log(x-i\delta)-\pi i)dx \underset{\delta\to 0}{\longrightarrow} -\int_{\varepsilon}^{1} f(x)(\log(x)+\pi i)dx.$$

(b) By also allowing ε to tend to zero, show that $\frac{1}{2\pi i}\int_{\gamma} f(z)(\log z - \pi i)dz = \int_0^1 f(x)dx$, where $\gamma(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$.

(c) Use that observation to prove the bound $\left|\int_0^1 f(x)dx\right| \le \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})d\theta\right|$ (you can prove a strong inequality as well).

9. By integrating along a keyhole curve, compute the following integrals:

(a) $\int_0^\infty \frac{\ln(x)^2}{1+x+x^2}dx$

(b) $\int_0^\infty \frac{\ln(1+x^2)}{x^{1+\alpha}}dx$, where $0 < \alpha < 2$ (hint: you might convert this integral into a more comfortable form)

(c) $\int_0^\infty x^\alpha \frac{p(x)}{q(x)}dx$, where $-1 < \alpha < 1$, p, q are polynomials with $\deg q \ge 2 + \deg p$, and q has no real zeroes. Here, the answer should naturally be given in terms of p, q.

10. (a) How many roots of the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ lie inside the unit disc?

(b) How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane?

(c) For how many z with $\mathrm{Re}\,(z) \ge 0$ is it true that $e^{iz} = z^2 + 2$?

(d) For any $x > 1$, show that there exists a unique solution to the equation $e^{-z} = x - z$ in the right half plain, and that this solution is real.

11. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, and $f(a) = w$ for some $a \in \Omega$. Suppose $g(z) = f(z) - b$ has a zero of order 2 at a. Show that there exists a disk $D_r(b)$ and an open set $a \in U$ such that for any c in the punctured disk $D'_r(b)$, the equation $f(z) = c$ has two *distinct* roots in U.

12. Let $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be defined as the angle between the segments $[0, x + iy]$ and $[x + iy, 1]$. Show that $u(x, y)$ is harmonic.